## Extra Sheet - Solutions

a) $\displaystyle\int \frac{e^t\,dt}{e^{2t}+3e^t+2} = \int \frac{du}{u^2+3u+2}$ , $[u = e^t,\ du = e^t dt]$

$\displaystyle = \int\left(\frac{1}{u+1} - \frac{1}{u+2}\right) du$ , P.F.D.

$= \ln|u+1| - \ln|u+2| + C = \ln(e^t+1) - \ln(e^t+2) + C$

b) $\displaystyle\int \frac{\cos y\,dy}{\sin^2 y + \sin y - 6} = \int \frac{du}{u^2+u-6}$ , $[u = \sin y,\ du = \cos y\,dy]$

$\displaystyle = \frac{1}{5}\int\left(\frac{1}{u-2} - \frac{1}{u+3}\right) d$ , P.F.D.

$\displaystyle = \frac{1}{5}\ln|u-2| - \frac{1}{5}\ln|u+3| + C$

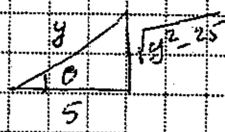
c) $\displaystyle\int \frac{\sqrt{y^2-25}\ dy}{y^3}$ , $\left[\begin{array}{l} y = 5\sec\theta,\ 0<\theta<\frac{\pi}{2} \\ dy = 5\sec\theta\tan\theta\,d\theta \\ \sqrt{y^2-25} = 5\tan\theta \end{array}\right]$

$\displaystyle = \int \frac{25\tan^2\theta\sec\theta\,d\theta}{125\sec^3\theta}$

$\displaystyle = \frac{1}{5}\int \frac{\tan^2\theta}{\sec^2\theta}\,d\theta = \frac{1}{5}\int \sin^2\theta\,d\theta = \frac{1}{10}\int(1-\cos 2\theta)\,d\theta$

$\displaystyle = \frac{1}{10}\left(\theta - \frac{\sin 2\theta}{2}\right) + c = \frac{1}{10}\left(\theta - \sin\theta\cos\theta\right) + c$

$\displaystyle = \frac{1}{10}\left[\sec^{-1}\frac{y}{5} - \left(\frac{\sqrt{y^2-25}}{y}\right)\left(\frac{5}{y}\right)\right] + c$ ,




$\displaystyle = \frac{1}{10}\sec^{-1}\frac{y}{5} - \frac{\sqrt{y^2-25}}{2y^2} + c$

d) $\displaystyle\int_0^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$ , $[u = e^t,\ du = e^t dt]$

$\displaystyle = \int_1^4 \frac{du}{\sqrt{u^2+9}} = \left[\sinh^{-1}\frac{u}{3}\right]_1^4 = \sinh^{-1}\frac{4}{3} - \sinh^{-1} 1$

(or use $u = 3\tan\theta$)

e) $\displaystyle\int \frac{\cot(3+\ln x)\,dx}{x}$ , $\left[u = 3+\ln x,\ du = \frac{dx}{x}\right.$

$\displaystyle = \int \cot u\,du = \ln|\sin u| + C = \ln|\sin(3+\ln x)| + C$

f) $\displaystyle\int \cot^3 y\csc^2 y\,dy$ , $[u = \cot y,\ du = -\csc^2 y\,dy]$

$\displaystyle = \int -u^3\,du = -\frac{u^4}{4} + c = -\frac{\cot^4 y}{4} + c$

g) $\displaystyle\int \sqrt{\frac{1-\cos x}{2}}\ dx = \int\sqrt{\sin^2\frac{x}{2}}\ dx = \int\left|\sin\frac{x}{2}\right| dx$

$\displaystyle = \begin{cases} -2\cos\frac{x}{2} + C & \text{if } \sin\frac{x}{2} \geq 0 \\ 2\cos\frac{x}{2} + c & \text{if } \sin\frac{x}{2} < 0 \end{cases}$

(2)

h) $\int \frac{dx}{1+\sin x} = \int \frac{1-\sin x}{(1+\sin x)(1-\sin x)} dx = \int \frac{1-\sin x}{1-\sin^2 x} dx$

$= \int \frac{1-\sin x}{\cos^2 x} dx = \int \frac{1}{\cos^2 x} dx - \int \frac{\sin x}{\cos^2 x} dx$

$= \int \sec^2 x \, dx - \int \sec x \tan x \, dx = \tan x - \sec x + c$

i) $\int \frac{dx}{\csc x + \cot x} = \int \frac{\sin x}{1+\cos x} dx$ , $[u = 1+\cos x, du = -\sin x \, dx]$

$= \int \frac{-du}{u} = -\ln|u| + c = -\ln|1+\cos x| + c$

j) $\int \frac{\ln x \, dx}{x + 4x \ln^2 x} = \int \frac{\ln x \, dx}{x(1+4\ln^2 x)}$ , $\left[u = \ln^2 x, du = \frac{2\ln x}{x} dx\right]$

$= \int \frac{du}{2(1+4u)} = \frac{1}{2}\int \frac{du}{1+4u}$ , $[y = 1+4u, dy = 4du]$

$= \frac{1}{8}\int \frac{dy}{y} = \frac{1}{8} \ln|y| + c = \frac{1}{8}\ln(1+4\ln^2 x) + c$

k) $\int (\csc x - \sec x)(\sin x + \cos x) dx$ [expand]

$= \int (1 + \cot x - \tan x - 1) dx = \int (\cot x - \tan x) dx$

$= \ln|\sin x| - \ln|\sec x| + c$

l) $\int \frac{8 \, dx}{x^2 - 2x + 2} = \int \frac{8 \, dx}{(x-1)^2 + 1}$ , $[u = x-1, du = dx]$

$= \int \frac{8 \, du}{u^2+1} = 8 \tan^{-1} u + c = 8 \tan^{-1}(x-1) + c$

m) $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}} = \int \frac{dx}{(x-2)\sqrt{(x-2)^2-1}}$ , $[u = x-2, du = dx]$

$= \int \frac{du}{u\sqrt{u^2-1}} = \sec^{-1}|u| + c = \sec^{-1}|x-2| + c$

n) $\int \sec^2 x \tan(\tan x) dx$ , $[u = \tan x, du = \sec^2 x \, dx]$

$= \int \tan u \, du = \ln|\sec u| + c = \ln|\sec(\tan x)| + c$

o) $\int e^{\sqrt{3x+9}} dx$ , $\left[u = \sqrt{3x+9}, du = \frac{3}{2\sqrt{3x+9}} dx \Rightarrow dx = \frac{2u \, du}{3}\right]$

$= \frac{2}{3}\int u e^u du$

$= \frac{2}{3}\left[u e^u - \int e^u du\right]$ , $\left[\begin{array}{l} w = u \Rightarrow dw = du \\ dV = e^u du \Rightarrow V = e^u \end{array}\right]$ Integrate by parts

$= \frac{2}{3}\sqrt{3x+9}\, e^{\sqrt{3x+9}} - \frac{2}{3} e^{\sqrt{3x+9}} + c$

p) $\int e^{2x}\cos 3x\,dx$ , $\left[\begin{array}{l} u = e^{2x} \Rightarrow du = 2e^{2x}dx \\ dv = \cos 3x\,dx \Rightarrow v = \frac{\sin 3x}{3} \end{array}\right]$

$= \frac{e^{2x}\sin 3x}{3} - \int \frac{2}{3} e^{2x}\sin 3x\,dx$ , $\left[\begin{array}{l} u = e^{2x} \Rightarrow du = 2e^{2x}dx \\ dV = \sin 3x\,dx \Rightarrow v = -\frac{\cos 3x}{3} \end{array}\right]$

$= \frac{e^{2x}\sin 3x}{3} - \frac{2}{3}\left(-\frac{e^{2x}\cos 3x}{3} - \int -\frac{2}{3} e^{2x}\cos 3x\,dx\right)$

Let $\int e^{2x}\cos 3x\,dx = I$

$\Rightarrow I = \frac{e^{2x}\sin 3x}{3} + \frac{2e^{2x}\cos 3x}{9} - \frac{4}{9} I$

$\Rightarrow I + \frac{4}{9} I = \frac{e^{2x}\sin 3x}{3} + \frac{2e^{2x}\cos 3x}{9}$

$\Rightarrow I = \frac{9}{13}\left(\frac{e^{2x}\sin 3x}{3} + \frac{2e^{2x}\cos 3x}{9}\right) + C$

$\Rightarrow I = \frac{3}{13} e^{2x}\sin 3x + \frac{2}{13} e^{2x}\cos 3x + C$

q) $\int x\sqrt{1-x}\,dx$ , $\left[\begin{array}{l} u = x \Rightarrow du = dx \\ dV = \sqrt{1-x}\,dx \Rightarrow V = -\frac{2}{3}(1-x)^{\frac{3}{2}} \end{array}\right]$

$= -\frac{2}{3}x(1-x)^{\frac{3}{2}} + \frac{2}{3}\int (1-x)^{\frac{3}{2}}dx$

$= -\frac{2x}{3}(1-x)^{\frac{3}{2}} - \frac{2}{3}\cdot\frac{2}{5}(1-x)^{\frac{5}{2}} + C$

$= -\frac{2x}{3}(1-x)^{\frac{3}{2}} - \frac{4}{15}(1-x)^{\frac{5}{2}} + C$

r) $\int \sin 2x\cos 3x\,dx = \int \frac{\sin(2x-3x) + \sin(2x+3x)}{2}\,dx$

$= \frac{1}{2}\int(\sin(-x) + \sin 5x)\,dx = \frac{1}{2}\left(\frac{-\cos(-x)}{-1} - \frac{\cos 5x}{5}\right) + C$

$= \frac{1}{2}\cos(-x) - \frac{1}{10}\cos 5x + C$

s) $\int \frac{\sqrt{x}\,dx}{\sqrt{1-x}} = \int \frac{\sqrt{x}\,dx}{\sqrt{1-(\sqrt{x})^2}}$ , $u = \sqrt{x}$ , $du = \frac{1}{2\sqrt{x}}dx \Rightarrow dx = 2u\,du]$

$= \int \frac{2u^2\,du}{\sqrt{1-u^2}}$ , $\left[\begin{array}{l} u = \sin\theta,\ du = \cos\theta\,d\theta \\ \sqrt{1-u^2} = \cos\theta \end{array}\right]$

$= \int 2\frac{\sin^2\theta\cdot\cos\theta\,d\theta}{\cos\theta} = \int(1-\cos 2\theta)\,d\theta = \theta - \frac{1}{2}\sin 2\theta + c$

$= \theta - \sin\theta\cos\theta + c = \sin^{-1}\sqrt{x} - \sqrt{x}\cdot\sqrt{1-x} + c$

t) $\int x^3 e^{\frac{x}{2}}dx$ Tabular integration

$$\left[\begin{array}{lcl} x^3 & + & e^{\frac{x}{2}} \\ 3x^2 & - & 2e^{\frac{x}{2}} \\ 6x & + & 4e^{\frac{x}{2}} \\ 6 & - & 8e^{\frac{x}{2}} \\ 0 & & 16e^{\frac{x}{2}} \end{array}\right]$$

$= 2x^3e^{\frac{x}{2}} - 12x^2e^{\frac{x}{2}} + 48xe^{\frac{x}{2}} - 96e^{\frac{x}{2}} + c$

u) $\int x^2\, 2^{-x}dx$

$= -\frac{x^2 2^{-x}}{\ln 2} - \frac{2x\,2^{-x}}{(\ln 2)^2} - \frac{2\cdot 2^{-x}}{(\ln 2)^3} + C$

$$\left[\begin{array}{lcl} x^2 & + & 2^{-x} \\ 2x & - & \frac{-2^{-x}}{\ln 2} \\ 2 & + & \frac{2^{-x}}{(\ln 2)^2} \\ 0 & & \frac{-2^{-x}}{(\ln 2)^3} \end{array}\right]$$